A-HYPERGEOMETRIC FUNCTIONS IN TRANSCENDENTAL QUESTIONS OF ALGEBRAIC GEOMETRY

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ABSTRACT. We generalize the known constructions of A-hypergeometric functions. In particular, we show that periods of middle dimension on affine or projective complex algebraic varieties are A-hypergeometric functions of coefficients of polynomial equations of these varieties.

1. Introduction.

In a series of papers, I. M. Gelfand with co-authors have introduced and studied the important class of A-hypergeometric functions. The definition of A-hypergeometric system of linear partial differential equations and a study of its solutions are given in [1].

The goal of this note is to provide a general algebro-geometric construction of A-hypergeometric functions including the known constructions as particular cases. Let us recall these known constructions.

a) In [2] it is shown that periods of products of complex powers of arbitrary polynomials of several variables, i. e. integrals

$$\oint_C f_1(x_1,\ldots,x_m)^{\lambda_1}\ldots f_n(x_1,\ldots,x_m)^{\lambda_n}x_1^{\beta_1-1}\ldots x_m^{\beta_m-1}dx_1\ldots dx_m \tag{1}$$

over an m-dimensional real cycle C with values in the corresponding local system, are A-hypergeometric functions of the coefficients of the polynomials $f_1,\ldots,f_n$.

b) In [3] it is shown that periods of exponent of an arbitrary polynomial, i. e. integrals

$$\int_C e^{f(x_1,\ldots,x_m)}x_1^{\beta_1-1}\ldots x_m^{\beta_m-1}dx_1\ldots dx_m, \tag{2}$$

where C is a possibly non-compact m-dimensional contour (with values in the local system) such that the expression under the integral tends on it to zero at infinity, are A-hypergeometric functions of the coefficients of the polynomial f.

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c) Recall the fundamental theorem of B. Sturmfels [4] which is the "constructive main theorem of algebra".

Theorem 1. *The complex roots of an arbitrary algebraic equation of one variable*

$$f(x) = 0 \tag{3}$$

form a (multivalued) A-hypergeometric function of the coefficients of the polynomial $f(x)$.

All these theorems are checked by direct differentiation showing that the required quantity (the integral or the root) satisfies the A-hypergeometric system of partial differential equations as a function of coefficients.

The present note arose in attempts to understand and to unify these constructions. The result is a general construction from the theory of periods of algebraic varieties, see the Main Theorem below. This theorem and its corollaries, Theorems 2, 3 below, show that the A-hypergeometric functions should play an important role in motivic constructions of algebraic geometry, bridging the gap between purely analytic and purely algebraic theories.

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2. Main theorem.

Theorem. *Let*

$$x = (x_1, \dots, x_m) \to z = f(x) = (f_1(x), \dots, f_n(x))$$

be a polynomial map of affine or projective spaces, where $f_i(x) = \sum_j a_{ij} x^j$, $1 \le i \le n$, $j = (j_1, \dots, j_m)$, $x^j = x_1^{j_1} \dots x_m^{j_m}$. Let $g = g(z_1, \dots, z_n)$ be a (multivalued in general) holomorphic function, which is quasi-homogeneous in the variables z_i (i. e. $z_i \partial g / \partial z_i = \alpha_i g$ for certain number α_i and for each i). Let ω be a (multivalued in general) holomorphic differential m-form of the variables x, which is quasi-homogeneous in the variables x_p (i. e. $x_p \partial \omega / \partial x_p = \beta_p \omega$). Then the integrals

$$\oint_C g(f(x))\omega \tag{4}$$

are A-hypergeometric functions of the coefficients a_{ij}. Here C is an m-dimensional cycle with values in the local system determined by the multivalued expression under the integral.

Proof. The A-hypergeometric PDE's follow from the following computations:

(5)
$$\frac{\partial^q}{\partial a_{i_1 j_{(1)}} \dots \partial a_{i_q j_{(q)}}} \oint_C g(f(x))\omega = \oint_C \frac{\partial^q g}{\partial z_{i_1} \dots \partial z_{i_q}} (f(x)) x^{j_{(1)}+\dots+j_{(q)}} \omega;$$

further, for each $p = 1, \dots, m$,

$$\begin{aligned} &\sum_{i,j} j_p a_{ij} \frac{\partial}{\partial a_{ij}} \oint_C g(f(x))\omega = \oint_C \sum_{i,j} \frac{\partial g}{\partial z_i}(f(x)) j_p a_{ij} x^j \omega \\ &= \oint_C x_p \frac{\partial g(f(x))}{\partial x_p} \omega = -\beta_p \oint_C g(f(x))\omega, \end{aligned} \tag{6}$$

since the form ω is quasi-homogeneous, and $g(f(x))\omega$ is closed; further, for each $i_0 = 1, \dots, n$,

$$\begin{aligned} &\sum_j a_{i_0 j} \frac{\partial}{\partial a_{i_0 j}} \oint_C g(f(x))\omega = \oint_C \sum_j \frac{\partial g}{\partial z_{i_0}} (f(x)) a_{i_0 j} x^j \omega \\ &= \oint_C \frac{\partial g}{\partial z_{i_0}} (f(x)) z_{i_0}(x) \omega = \alpha_{i_0} \oint_C g(f(x))\omega, \end{aligned} \tag{7}$$

since the function $g(z)$ is quasi-homogeneous. □

3. Examples. Example a) from the Introduction follows directly from the Main theorem if we put

$$g(z_1, \dots, z_n) = z_1^{\lambda_1} \dots z_n^{\lambda_n}, \quad \omega = x_1^{\beta_1 - 1} \dots x_m^{\beta_m - 1} dx_1 \wedge \dots \wedge dx_m.$$

Example b) follows by putting $n = 1$,

$$g(z) = e^z, \quad \omega = x_1^{\beta_1 - 1} \dots x_m^{\beta_m - 1} dx_1 \wedge \dots \wedge dx_m.$$

In this case, equation (7) is not used (for equality (5) in this case implies more equations), so that $g(z)$ does not need to be quasi-homogeneous.

Example c) follows by putting $m = n = 1$, $g(z) = \frac{1}{2\pi i} \log z$, $\omega = dx$. Indeed, integrating by parts, we have $\oint \log z \, dx = -\oint x dz/z$. Equation (7) is derived in the same way.

d) **Theorem 2.** *Let*

$$f(x_1, \dots, x_k, y) = \sum c_{i_1 \dots i_k j} x_1^{i_1} \dots x_k^{i_k} y^j = 0 \tag{8}$$

be an arbitrary polynomial equation. Then y is a (multivalued) A-hypergeometric function of coefficients $c_{i_1 \dots i_k j}$, depending on $x_1, \dots, x_k$ as on parameters.

This theorem is obtained if we put $n = 1$, $g(z) = \log z$, $\omega = dy$ in the Main Theorem. In this case, the quasi-homogeneity equation (6) is used only with respect to the variable y, and $x_1,\dots,x_k$ play the role of

parameters, so that the form $g(f(x,y))\omega$ needs to be closed only with respect to y.

e) The following theorem is obtained if we put

$$g(z_1,\ldots,z_n) = z_1^{\lambda_1}\ldots z_l^{\lambda_l}/(z_{l+1}\ldots z_n)$$

in the Main Theorem and use the Cauchy residue theorem.

Theorem 3. *The Gelfand–Leray integral*

$$\oint_C f_1(x)^{\lambda_1}\ldots f_l(x)^{\lambda_l} x_1^{\beta_1-1}\ldots x_m^{\beta_m-1} \\ dx_1\wedge\ldots\wedge dx_m/(df_{l+1}\wedge\ldots\wedge df_n)|_{f_{l+1}(x)=\ldots=f_n(x)=0} \tag{9}$$

over a cycle C with values in the corresponding local system on the variety

$$f_{l+1}(x_1,\ldots,x_m) = \ldots = f_n(x_1,\ldots,x_m) = 0, \tag{10}$$

is an A-hypergeometric function of the coefficients of the polynomials f_1, ..., f_n.

In the case $m=n$, $l=0$, this Theorem is stated in [4].

Remark. All the results of this paper remain valid if one replaces affine or projective spaces by arbitrary toric varieties.

References

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